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# A Finite Algorithm for Globally Optimizing a Class of Rank-Two Reverse Convex Programs

TAKAHITO KUNO[1,*] and YOSHITSUGU YAMAMOTO[2]
[1]*Institute of Information Sciences and Electronics, University of Tsukuba, Tsukuba, Ibaraki 305, Japan (email: takahito@is.tsukuba.ac.jp)*
[2]*Institute of Policy and Planning Sciences, University of Tsukuba, Tsukuba, Ibaraki 305, Japan (email: yamamoto@shako.sk.tsukuba.ac.jp)*

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**Abstract.** In this paper, we propose an algorithm for solving a linear program with an additional rank-two reverse convex constraint. Unlike the existing methods which generate approximately optimal solutions, the algorithm provides a rigorous optimal solution to this nonconvex problem by a finite number of dual pivot operations. Computational results indicate that the algorithm is practical and can solve fairly large scale problems.

**Key words:** Global optimization, Reverse convex program, Rank-two quasiconcave function, Parametric simplex algorithm.

## 1. Introduction

In this paper, we describe a method for solving a special class of reverse convex programs [5, 21]:

$$\text{maximize}\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x} \in X \cap Y\}, \tag{1.1}$$

where $\boldsymbol{c} \in \mathbb{R}^n$, and $X \subseteq \mathbb{R}^n$ is a polytope. The reverse convex set $Y \subseteq \mathbb{R}^n$ is defined below by a function $f : \mathbb{R}^n \to \mathbb{R}$, which is strictly quasiconcave and has rank-two monotonicity on an open convex set $X^\circ$ including $X$:

$$Y = \{\boldsymbol{x} \in X^\circ \mid f(\boldsymbol{x}) \leq 0\}.$$

Since $Y$ is the complement of a convex set $\{\boldsymbol{x} \in X^\circ \mid f(\boldsymbol{x}) > 0\}$ relative to $X^\circ$, the feasible region might be neither convex nor connected. Therefore, the objective function of (1.1) can have multiple local maxima in $X \cap Y$, many of which fail to be globally optimal. The detailed definition of rank-two monotonicity will be given in section 2 (see also [12, 18, 23]).

A typical example of (1.1) is a linear program with an additional linear multiplicative constraint [15, 20, 24]:

$$\text{maximize}\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x} \in X,\ (\boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x} + d_{10})(\boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x} + d_{20}) - d_{00} \leq 0\}, \tag{1.2}$$

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where $\boldsymbol{d}_i \in \mathbb{R}^n$, $i = 1, 2$, $d_{i0} \in \mathbb{R}$, $i = 0, 1, 2$, and $X$ is assumed to be included in $X^\circ = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{d}_i^{\mathrm{T}}\boldsymbol{x} + d_{i0} > 0,\ i = 1, 2\}$. The product of two affine functions appears in many applications such as microeconomics [4], bond portfolio optimization [8] and geometrical optimization [11, 14] and so forth (see [10, 17]). In [15, 24], we proposed a branch-and-bound algorithm for generating an $\epsilon$-optimal solution. We reduced (1.2) to a problem of minimizing a univariate function, whose values we computed by solving convex programs. In [16], we extended this idea and solved more general class of problems than (1.2). In [20], Thach et al. converted (1.2) into a two-dimensional concave minimization problem and applied an outer approximation algorithm.

As regards the problem (1.1), Pferschy and Tuy developed a promising algorithm to generate an $\epsilon$-optimal solution in [18]. Their algorithm based on an approach in [21] consists mainly of two procedures: the first one moves from vertex to vertex along edges of $X$ and finds a local maximum $\boldsymbol{x}'$; the second one checks the $\epsilon$-optimality of $\boldsymbol{x}'$ by minimizing the constraint function $f$. Due to the rank-two monotonicity, one can minimize $f$ very efficiently using any one of parametric simplex algorithms, e.g. proposed in [9, 13, 23]. If $\boldsymbol{x}'$ turns out not to be an $\epsilon$-optimal solution, a cutting plane constraint $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \geq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}' + \epsilon$ is added to exclude those points with objective function values less than $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}' + \epsilon$. Our algorithm contrasts with the method by Pferschy and Tuy in two points: using no cutting planes and yielding a globally optimal solution within finitely many steps.

The organization of the paper is as follows. In section 2, we parametrize (1.1) by introducing a vector $\boldsymbol{\xi}$ of two auxiliary variables. We show that an optimal solution to the resulting linear program solves (1.1) only if $\boldsymbol{\xi}$ lies in some set $\Xi^*$ associated with the boundaries of $X$ and $Y$. In section 3, to search each connected subset of $\Xi^*$, we apply a parametric dual simplex algorithm to the linear program. In section 4, using this algorithm as a procedure, we locate a point providing a globally optimal solution to (1.1) in the whole of $\Xi^*$. Computational results of the algorithm are reported in section 5.

## 2. Parametrization of the problem

The nonconvex program we consider in this paper is

$$
[\mathrm{P}]\left|\begin{array}{ll} \text{maximize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \mathbf{0}, \\ & f(\boldsymbol{x}) \leq 0, \end{array}\right.
$$

where $A \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{c} \in \mathbb{R}^n$. We assume that

$$
X = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = b,\ \boldsymbol{x} \geq \mathbf{0}\}
$$

is a nonempty and bounded subset of an open convex set $X^\circ \subseteq \mathbb{R}^n$. The constraint function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and strictly quasiconcave on $X^\circ$, i.e. for each $\boldsymbol{x}, \boldsymbol{y} \in X^\circ$ with $f(\boldsymbol{x}) \neq f(\boldsymbol{y})$ we have

$$f((1-\lambda)\boldsymbol{x} + \lambda \boldsymbol{y}) > \min\{f(\boldsymbol{x}), f(\boldsymbol{y})\} \text{ for any } \lambda \in (0,1). \tag{2.1}$$

We also assume $f$ to possess rank-two monotonicity on $X^\circ$ with respect to linearly independent vectors $\boldsymbol{d}_1, \boldsymbol{d}_2 \in \mathbb{R}^n$ [12, 18, 23]. Namely, for each $\boldsymbol{x}, \boldsymbol{y} \in X^\circ$,

$$\boldsymbol{d}_i^{\mathrm{T}}\boldsymbol{x} \leq \boldsymbol{d}_i^{\mathrm{T}}\boldsymbol{y} \text{ for } i = 1, 2 \text{ implies that } f(\boldsymbol{x}) \leq f(\boldsymbol{y}). \tag{2.2}$$

Let

$$Y = \{\boldsymbol{x} \in X^\circ \mid f(\boldsymbol{x}) \leq 0\}.$$

The feasible region of [P], denoted by $X \cap Y$, is the difference of a polytope $X$ and an open convex set $X^\circ \setminus Y$. If we remove the last constraint $f(\boldsymbol{x}) \leq 0$, we have an ordinary linear program:

$$[\bar{\mathrm{P}}] \quad \text{maximize}\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x} \in X\},$$

which has an optimal solution $\bar{\boldsymbol{x}}$ because $X$ is nonempty and bounded. If $\bar{\boldsymbol{x}} \in Y$, then $\bar{\boldsymbol{x}}$ is globally optimal to [P]. To exclude such a trivial case, we assume throughout the paper that

$$\max\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x} \in X\} > \max\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x} \in X \cap Y\}. \tag{2.3}$$

REMARK. Condition (2.3) can be checked easily. Let $\bar{X} = X \cap \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{c}^{\mathrm{T}}\bar{\boldsymbol{x}}\}$. Then $\bar{X}$ contains no points satisfying $f(\boldsymbol{x}) \leq 0$ if and only if (2.3) holds. Therefore, we have only to minimize $f(\boldsymbol{x})$ over $\bar{X}$. Due to the rank-two monotonicity of $f$, this can be done by parametrically solving

$$\left|\begin{array}{ll} \text{minimize} & (1-\lambda)\boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x} + \lambda \boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{c}^{\mathrm{T}}\bar{\boldsymbol{x}},\ \boldsymbol{x} \geq \boldsymbol{0}, \end{array}\right.$$

and evaluating $f$ at the vertices encountered (see [23] for further details).

Let us denote by $\partial Y$ the set of boundary points of $Y$ relative to the topology induced on $X^\circ$. Since $f$ is continuous and strictly quasiconcave, the level surface $L_0 = \{\boldsymbol{x} \in X^\circ \mid f(\boldsymbol{x}) = 0\}$ coincides with either $\partial Y$ or the upper level set $L_+ = \{\boldsymbol{x} \in X^\circ \mid f(\boldsymbol{x}) \geq 0\}$ (see e.g. Proposition 3.31 in [1]). If $L_0 = L_+$, then

$$\begin{aligned} Y = \{\boldsymbol{x} \in X^\circ \mid f(\boldsymbol{x}) \leq 0\} &= L_0 \cup \{\boldsymbol{x} \in X^\circ \mid f(\boldsymbol{x}) < 0\} \\ &= L_+ \cup \{\boldsymbol{x} \in X^\circ \mid f(\boldsymbol{x}) < 0\} = X^\circ, \end{aligned}$$

which contradicts (2.3). Hence, we have

$$\partial Y = \{\boldsymbol{x} \in X^\circ \mid f(\boldsymbol{x}) = 0\}. \tag{2.4}$$

We also denote by $S(X)$ the one-dimensional skeleton of $X$, i.e. the union of edges and vertices of $X$. Under condition (2.3), we have the following theorem, which holds for linear programs with a general reverse convex constraint as well:

THEOREM 2.1. *If $X \cap Y \neq \emptyset$, then $X \cap \partial Y$ contains all globally optimal solutions to [P], at least one of which lies on $S(X) \cap \partial Y$.*

*Proof.* Follows from Corollary 2.1 in Tuy [21] and Proposition IX.11 in Horst and Tuy [7] (see also [6, 22]). □

The vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ characterizing the constraint function $f$ transform $X$ and $X^\circ$ respectively into

$$\Xi = \{(\boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x}, \boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x}) \mid \boldsymbol{x} \in X\}, \quad \Xi^\circ = \{(\boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x}, \boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x}) \mid \boldsymbol{x} \in X^\circ\}.$$

In the space of $\Xi^\circ$, we can have an insight into the rank-two monotonicity of $f$.

LEMMA 2.2. *There exists a function $g : \mathbb{R}^2 \to \mathbb{R}$ which is continuous, strictly quasiconcave on $\Xi^\circ$ and satisfies the following:*

$$f(\boldsymbol{x}) = g(\boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x}, \boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x}) \text{ for } \boldsymbol{x} \in X^\circ, \tag{2.5}$$

$$g(\boldsymbol{\xi}) \leq g(\boldsymbol{\eta}) \text{ if } \boldsymbol{\xi}, \boldsymbol{\eta} \in \Xi^\circ \text{ and } \boldsymbol{\xi} \leq \boldsymbol{\eta}. \tag{2.6}$$

*Proof.* If $f$ is not expressed as (2.5), there are two distinct points $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ in $X^\circ$ such that

$$\boldsymbol{d}_i^{\mathrm{T}}\boldsymbol{x}^1 = \boldsymbol{d}_i^{\mathrm{T}}\boldsymbol{x}^2, \quad i = 1, 2; \quad f(\boldsymbol{x}^1) \neq f(\boldsymbol{x}^2).$$

We may assume without loss of generality that $f(\boldsymbol{x}^1) < f(\boldsymbol{x}^2)$. Then it follows from (2.2) that

$$\exists i, \quad \boldsymbol{d}_i^{\mathrm{T}}\boldsymbol{x}^1 < \boldsymbol{d}_i^{\mathrm{T}}\boldsymbol{x}^2,$$

which is a contradiction. Hence, (2.5) holds for some function $g : \mathbb{R}^2 \to \mathbb{R}$.

Let $\boldsymbol{\xi}, \boldsymbol{\eta} \in \Xi^\circ$. Also, let $\boldsymbol{x}$ and $\boldsymbol{y}$ be points in $X^\circ$ satisfying $\boldsymbol{\xi} = (\boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x}, \boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x})$ and $\boldsymbol{\eta} = (\boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{y}, \boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{y})$, respectively. If $\boldsymbol{\xi} \leq \boldsymbol{\eta}$, then

$$g(\boldsymbol{\xi}) = f(\boldsymbol{x}) \leq f(\boldsymbol{y}) = g(\boldsymbol{\eta})$$

and (2.6) is yielded. If $g(\boldsymbol{\xi}) < g(\boldsymbol{\eta})$, by the strict quasiconcavity of $f$ we have

$$g((1-\lambda)\boldsymbol{\xi} + \lambda\boldsymbol{\eta}) = f((1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y}) > f(\boldsymbol{x}) = g(\boldsymbol{\xi}), \quad \forall\lambda \in (0, 1),$$

which implies the strict quasiconcavity of $g$ on $\Xi^\circ$. The continuity of $g$ can easily be checked. □

By exploiting the function $g$ and by introducing a vector $\boldsymbol{\xi}$ of two auxiliary variables $\xi_1$ and $\xi_2$, we can transform [P] into an equivalent form:

$$\text{[MP]} \left| \begin{array}{ll} \text{maximize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & \boldsymbol{x} \in X, \quad\quad g(\boldsymbol{\xi}) \leq 0, \\ & \boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x} = \xi_1, \ \boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x} = \xi_2. \end{array} \right.$$

The following is an immediate consequence:

LEMMA 2.3. *If* $(\boldsymbol{x}^*, \boldsymbol{\xi}^*)$ *is an optimal solution to [MP], then* $\boldsymbol{x}^*$ *solves [P].*

Let

$$\mathrm{H} = \{\boldsymbol{\xi} \in \Xi^\circ \mid g(\boldsymbol{\xi}) \le 0\},$$

and let $\partial\mathrm{H}$ denote the boundary of H relative to $\Xi^\circ$. In the same way as we have seen for $\partial Y$, the strict quasiconcavity of $g$ leads to

$$\partial\mathrm{H} = \{\boldsymbol{\xi} \in \Xi^\circ \mid g(\boldsymbol{\xi}) = 0\}.$$

Note that the slope of the tangent to $\partial\mathrm{H}$ is always nonpositive by the monotonicity property (2.6). We also see for $\boldsymbol{x} \in X^\circ$ that $\boldsymbol{x} \in \partial Y$ if and only if $\boldsymbol{\xi} = (\boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x}, \boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x}) \in \partial\mathrm{H}$. If we fix the values of $\xi_1$ and $\xi_2$ in [MP], we have a linear program:

$$[\mathrm{P}(\boldsymbol{\xi})]\left|\begin{array}{ll} \text{maximize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & \boldsymbol{x} \in X, \\ & \boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x} = \xi_1,\ \boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x} = \xi_2. \end{array}\right.$$

We refer to $\boldsymbol{\xi}$ as an *active point* if $[\mathrm{P}(\boldsymbol{\xi})]$ is feasible and $\boldsymbol{\xi}$ lies on $\partial\mathrm{H}$. Let $\Xi^* = \Xi \cap \partial\mathrm{H}$ and let $\boldsymbol{x}^*(\boldsymbol{\xi})$ be an optimal solution to $[\mathrm{P}(\boldsymbol{\xi})]$ if $\boldsymbol{\xi} \in \Xi$. Then the observations made so far are summarized into the following:

THEOREM 2.4. *Let* $\boldsymbol{x}^* = \boldsymbol{x}^*(\boldsymbol{\xi}^*)$ *be a point which maximizes* $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^*(\boldsymbol{\xi})$ *over all* $\boldsymbol{\xi} \in \Xi^*$. *Then* $\boldsymbol{x}^*$ *is a globally optimal solution to [P].*

Problem [P] can therefore be solved if we solve the linear program $[\mathrm{P}(\boldsymbol{\xi})]$ as varying $\boldsymbol{\xi}$ over all active points. This could be done rather easily if the curve $\partial\mathrm{H}$ is parametrized by a single parameter, e.g. an explicit function $\psi$ such that $\xi_2 = \psi(\xi_1)$ is known for $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \partial\mathrm{H}$. However, such a favorable situation is not expected in general. What is even worse, the set $\Xi^*$ of active points may not be connected.

In the rest of the paper, we impose a nondegeneracy assumption for the sake of simplicity.

ASSUMPTION 2.1. Problem [P] satisfies the following three conditions:

(i) Matrix $A$ has full rank. Any subset of columns of $[A, \boldsymbol{b}]$ has full rank if the corresponding submatrix of $A$ has.

(ii) Any submatrix of $[A^{\mathrm{T}}, \boldsymbol{c}, \boldsymbol{d}_1, \boldsymbol{d}_2]$ has full rank if the corresponding submatrix of $A^{\mathrm{T}}$ has.

(iii) No vertices of $X$ are boundary points of $Y$.

Condition (i) implies that the polytope $X$ has no degenerate vertices; condition (ii) implies that $[\mathrm{P}(\boldsymbol{\xi})]$ has a unique optimal solution $\boldsymbol{x}^*(\boldsymbol{\xi})$ if it exists. We also see from Theorem 2.1 that no vertices of $X$ are optimal to [P] under condition (iii).

## 3. Search for a locally best active point

We have seen from Theorems 2.1 and 2.4 that a globally optimal solution $\boldsymbol{x}^*$ to [P] will be found if we enumerate all $\boldsymbol{\xi} \in \Xi^*$ such that $\boldsymbol{x}^*(\boldsymbol{\xi}) \in S(X)$. To state this systematically, let us observe the relationship between the active points and the skelton of $X$ a little more fully.

Let

$$\tilde{A} = \begin{bmatrix} A \\ \boldsymbol{d}_1^{\mathrm{T}} \\ \boldsymbol{d}_2^{\mathrm{T}} \end{bmatrix}, \quad \tilde{\boldsymbol{b}} = \begin{bmatrix} \boldsymbol{b} \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{e}^1 = \begin{bmatrix} \boldsymbol{0} \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{e}^2 = \begin{bmatrix} \boldsymbol{0} \\ 0 \\ 1 \end{bmatrix}.$$

Given an active point $\boldsymbol{\xi}^0$, let us consider the linear program

$$[\mathrm{P}(\boldsymbol{\xi}^0)] \left| \begin{array}{ll} \text{maximize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & \tilde{A}\boldsymbol{x} = \tilde{\boldsymbol{b}} - \boldsymbol{e}^1\xi_1^0 - \boldsymbol{e}^2\xi_2^0, \ \boldsymbol{x} \geq \boldsymbol{0}. \end{array} \right.$$

Let $B_0 \in \mathbb{R}^{(m+2)\times(m+2)}$ be an optimal basis matrix and let

$$[B_0, N_0] = \tilde{A}, \quad \begin{bmatrix} \boldsymbol{c}_B \\ \boldsymbol{c}_N \end{bmatrix} = \boldsymbol{c}, \quad \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{bmatrix} = \boldsymbol{x}$$

denote the corresponding partitioned matrix and vectors. We then have an optimal dictionary of $[\mathrm{P}(\boldsymbol{\xi}^0)]$:

$$\left| \begin{array}{l} \boldsymbol{x}_B = \bar{\boldsymbol{b}} - \bar{\boldsymbol{e}}^1\xi_1^0 - \bar{\boldsymbol{e}}^2\xi_2^0 - \bar{N}_0\boldsymbol{x}_N \\ z = \boldsymbol{c}_B^{\mathrm{T}}(\bar{\boldsymbol{b}} - \bar{\boldsymbol{e}}^1\xi_1^0 - \bar{\boldsymbol{e}}^2\xi_2^0) + \bar{\boldsymbol{c}}_N^{\mathrm{T}}\boldsymbol{x}_N, \end{array} \right. \tag{3.1}$$

where

$$\bar{N}_0 = B_0^{-1}N_0, \quad \bar{\boldsymbol{b}} = B_0^{-1}\tilde{\boldsymbol{b}}, \quad \bar{\boldsymbol{c}}_N^{\mathrm{T}} = (\boldsymbol{c}_N^{\mathrm{T}} - \boldsymbol{c}_B^{\mathrm{T}}\bar{N}_0), \quad \bar{\boldsymbol{e}}^i = B_0^{-1}\boldsymbol{e}^i, \quad i = 1, 2.$$

Note on dictionary (3.1) that at most one component of $\bar{\boldsymbol{b}} - \bar{\boldsymbol{e}}^1\xi_1^0 - \bar{\boldsymbol{e}}^2\xi_2^0$ is zero and the rest are positive by Assumption 2.1.

As is well known (see e.g. [2, 3]), the basis $B_0$ remains optimal to $[\mathrm{P}(\boldsymbol{\xi})]$ as long as $\boldsymbol{\xi}$ satisfies $\bar{\boldsymbol{b}} - \bar{\boldsymbol{e}}^1\xi_1 - \bar{\boldsymbol{e}}^2\xi_2 \geq \boldsymbol{0}$. Let

$$\Phi_0 = \{\boldsymbol{\xi} \in \mathbb{R}^2 \mid \bar{\boldsymbol{e}}^1\xi_1 + \bar{\boldsymbol{e}}^2\xi_2 \leq \bar{\boldsymbol{b}}\}.$$

Then $\Phi_0$ is polyhedral and bounded, since for any $\boldsymbol{\xi} \in \Phi_0$ we have

$$\min\{\boldsymbol{d}_i^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x} \in X\} \leq \xi_i \leq \max\{\boldsymbol{d}_i^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x} \in X\}, \quad i = 1, 2.$$

Moreover, $\Phi_0$ has a nonempty interior and hence is of two-dimension even if (3.1) is degenerate. In fact, if the $s$th component of $\bar{\boldsymbol{b}} - \bar{\boldsymbol{e}}^1\xi_1^0 - \bar{\boldsymbol{e}}^2\xi_2^0$ is zero, then for a sufficiently small $\delta > 0$ we have

$$\bar{\boldsymbol{e}}^1(\xi_1^0 - \delta\bar{e}_s^1) + \bar{\boldsymbol{e}}^2(\xi_2^0 - \delta\bar{e}_s^2) < \bar{\boldsymbol{b}}.$$

Between the polygon $\Phi_0$ and a two-dimensional face of $X$ exists a one-to-one correspondence. Let

$$F_0 = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = \boldsymbol{x}^*(\boldsymbol{\xi}),\ \boldsymbol{\xi} \in \Phi_0\}.$$

We immediately see that $F_0$ is polyhedral and bounded since it is the image of $\Phi_0$ under a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^n$. We can further show the following:

LEMMA 3.1. *Polytope $F_0$ is a two-dimensional face of $X$.*

*Proof*. For each $\boldsymbol{\xi} \in \Phi_0$, the optimal solution $\boldsymbol{x}^*(\boldsymbol{\xi})$ to [P($\boldsymbol{\xi}$)] lies on the intersection of $n-2$ hyperplanes defined by $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x}_N = \mathbf{0}$. (Note that $\boldsymbol{x}_B \in \mathbb{R}^{m+2}$ and $\boldsymbol{x}_N \in \mathbb{R}^{n-m-2}$.) This, together with $\boldsymbol{x}^*_B(\boldsymbol{\xi}) \geq \mathbf{0}$, implies that $F_0$ is a face of $X$ with dimensionality two at most. However, $\boldsymbol{x}^*_B(\boldsymbol{\xi}) = \bar{\boldsymbol{b}} - \bar{\boldsymbol{e}}^1\xi_1 - \bar{\boldsymbol{e}}^2\xi_2 > \mathbf{0}$ for $\boldsymbol{\xi} \in \mathrm{int}\Phi_0$, and besides $\bar{\boldsymbol{e}}^1$ and $\bar{\boldsymbol{e}}^2$ are linearly independent. We then conclude that $\dim F_0 = 2$. □

We refer to $\Phi_0$, a polyhedral subset of $\Xi$, as a *cell* of $\Xi$ associated with the basis $B_0$. Obviously, $\boldsymbol{\xi}$ is a vertex of $\Phi_0$ if and only if $\boldsymbol{x}^*(\boldsymbol{\xi})$ is a vertex of $F_0$. This implies that each $\boldsymbol{\xi} \in S(\Phi_0) \cap \partial\mathrm{H}$ provides a candidate $\boldsymbol{x}^*(\boldsymbol{\xi}) \in S(X) \cap \partial Y$ for an optimal solution to [P].

### 3.1. Generation of a sequence of active points

Let us proceed to the procedure for generating a sequence of active points $\boldsymbol{\xi}^1$, $\boldsymbol{\xi}^2$, ..., each of which satisfies $\boldsymbol{x}^*(\boldsymbol{\xi}^i) \in S(X)$. For an interval $\Omega$ of real numbers let

$$\Xi(\Omega) = \Xi \cap \{\boldsymbol{\xi} \in \Xi^\circ \mid \xi_1 \in \Omega\}. \tag{3.2}$$

The procedure starts from a given active point $\boldsymbol{\xi}^1 \in S(\Phi_0) \cap \partial\mathrm{H}$ and visits distinct $\boldsymbol{\xi}^j$s successively in $\Xi([\xi_1^1, \bar{\omega}]) \cap \partial\mathrm{H}$ for some number $\bar{\omega} \geq \xi_1^1$. The way to obtain a starting active point $\boldsymbol{\xi}^1$ will be discussed in the next section.

Since the cell $\Phi_0$ is a convex polygon defined by $m+2$ half planes, we can generate all the vertices in time $O(m \log m)$ using computational geometry (see e.g. [19]). Let $\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^p, \boldsymbol{\eta}^{p+1}$ $(= \boldsymbol{\eta}^1)$ denote the vertices of $\Phi_0$ in counterclockwise order from $\boldsymbol{\xi}^1$. Suppose the edge $\boldsymbol{\eta}^p$–$\boldsymbol{\eta}^1$ contains a point in $\Xi^\circ \setminus \mathrm{H}$. Then we have either of the following under condition (iii) of Assumption 2.1:

*case 3.1:* $g((1-\lambda)\boldsymbol{\eta}^p + \lambda\boldsymbol{\xi}^1) < 0$ for any $\lambda \in [0, 1)$;

*case 3.2:* $g((1-\lambda)\boldsymbol{\xi}^1 + \lambda\boldsymbol{\eta}^1) < 0$ for any $\lambda \in (0, 1]$.

In case 3.2, moving along $S(\Phi_0)$ counterclockwise from $\boldsymbol{\eta}^1$, we choose as $\boldsymbol{\xi}^2$ the last point where the value of $g$ is nonpositive. Let $\boldsymbol{\eta}^k$–$\boldsymbol{\eta}^{k+1}$ be the edge containing $\boldsymbol{\xi}^2$. Then

$$g((1-\lambda)\boldsymbol{\eta}^i + \lambda\boldsymbol{\eta}^{i+1}) \leq 0,\quad \forall\lambda \in [0, 1],\quad i = 1, \ldots, k-1,$$

and for $\boldsymbol{\eta}^k$–$\boldsymbol{\xi}^2$–$\boldsymbol{\eta}^{k+1}$ we have

$$g((1-\lambda)\boldsymbol{\eta}^k + \lambda\boldsymbol{\xi}^2) < 0, \quad \forall\lambda \in [0, 1),$$

just as in case 3.1 for $\boldsymbol{\eta}^p$–$\boldsymbol{\xi}^1$–$\boldsymbol{\eta}^1$. The active point $\boldsymbol{\xi}^2$ newly found satisfies $\xi_1^2 \geq \xi_1^1$ and $\xi_2^2 \leq \xi_2^1$, but is never equal to $\boldsymbol{\xi}^1$ because $\Phi_0$ is of two-dimension.

LEMMA 3.2. *In case 3.2, no $\boldsymbol{x}^*(\boldsymbol{\xi}) \in S(X)$ with $\boldsymbol{\xi}$ lying on $\partial$H between $\boldsymbol{\xi}^1$ and $\boldsymbol{\xi}^2$ can be optimal to [P], except for $\boldsymbol{x}^*(\boldsymbol{\xi}^1)$ and $\boldsymbol{x}^*(\boldsymbol{\xi}^2)$.*

*Proof.* If an edge $\boldsymbol{\eta}^q$–$\boldsymbol{\eta}^{q+1}$ ($k < q < p$) intersects $\partial$H between $\boldsymbol{\xi}^1$ and $\boldsymbol{\xi}^2$, the line segment $\boldsymbol{\xi}^1$–$\boldsymbol{\xi}^2$ does not entirely lie in $\Phi_0$, which contradicts the convexity of $\Phi_0$. This piece of $\partial$H is therefore included in $\Phi_0$ and has intersections with only the edges $\boldsymbol{\eta}^p$–$\boldsymbol{\eta}^1$ and $\boldsymbol{\eta}^i$–$\boldsymbol{\eta}^{i+1}$, $i = 1, \ldots, k$. Suppose $\boldsymbol{\eta}^r$–$\boldsymbol{\eta}^{r+1}$ ($1 \leq r < k$) touches $\partial$H at $\boldsymbol{\xi}'$. Then, by Assumption 2.1 (iii), we have

$$g(\boldsymbol{\eta}^r) < 0, \quad g(\boldsymbol{\eta}^{r+1}) < 0, \quad g(\boldsymbol{\xi}') = 0.$$

We see from Lemma 3.1 that $\boldsymbol{x}^*(\boldsymbol{\xi}')$ lies on an edge connecting two vertices $\boldsymbol{x}^*(\boldsymbol{\eta}^r)$ and $\boldsymbol{x}^*(\boldsymbol{\eta}^{r+1})$ of $X$. Both the vertices, however, lie in int$Y$, and hence neither is optimal to [P] by Theorem 2.1. Since $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^*(\boldsymbol{\xi}') \leq \max\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^*(\boldsymbol{\eta}^r), \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^*(\boldsymbol{\eta}^{r+1})\}$ holds, $\boldsymbol{x}^*(\boldsymbol{\xi}')$ is not optimal, either. □

Let us now turn to case 3.1. If we replace $\boldsymbol{\xi}^0$ by $\boldsymbol{\xi}^1$ in dictionary (3.1), then for the $s$th row corresponding to $\boldsymbol{\eta}^p$–$\boldsymbol{\eta}^1$ we have

$$\bar{b}_s - \bar{e}_s^1\xi_1^1 - \bar{e}_s^2\xi_2^1 = 0.$$

Selecting a variable to enter the basis appropriately from nonbasic variables and performing a single dual pivot operation, we obtain an alternative basis matrix $B_1$, which is also optimal to [P($\boldsymbol{\xi}^1$)]. The cell $\Phi_1$ associated with $B_1$ shares the edge $\boldsymbol{\eta}^p$–$\boldsymbol{\eta}^1$ with $\Phi_0$. Therefore, the rest of the procedure is the same as in case 3.2. If we cannot find any entering variables, i.e.

$$(\boldsymbol{e}^s)^{\mathrm{T}}\bar{N}_0 \geq \mathbf{0}, \tag{3.3}$$

then $\boldsymbol{x}^*(\boldsymbol{\xi}^1)$ is a maximum point of $\lambda_1\boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x} + \lambda_2\boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x}$ over $X$, where $\lambda_1 = \eta_2^1 - \eta_2^p$ and $\lambda_2 = \eta_1^p - \eta_1^1$. In other words, the edge $\boldsymbol{\eta}^p$–$\boldsymbol{\eta}^1$ determines a supporting line of $\Xi = \{(\boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x}, \boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x}) \mid \boldsymbol{x} \in X\}$; and $\Xi$ is included in

$$\Lambda = \{(\boldsymbol{\xi} \in \Xi^\circ \mid (\lambda_1, \lambda_2)(\boldsymbol{\xi} - \boldsymbol{\xi}^1) \leq 0\}.$$

LEMMA 3.3. *Suppose (3.3) holds in case 3.1. Then*
*(i)* $\Xi((\xi_1^1, +\infty)) \cap \partial\mathrm{H} = \emptyset$ *if* $\eta_1^1 \leq \eta_1^p$ *(and* $\eta_2^1 \geq \eta_2^p$*);*
*(ii)* $\Xi((\xi_1^1, \xi + \delta]) \cap \mathrm{H} = \emptyset$ *for some* $\delta > 0$ *otherwise.*

*Proof.* (i) Suppose $\eta_1^1 < \eta_1^p$; otherwise, the assertion is obvious. In case 3.1, we have $\Xi((\xi_1^1, \eta_1^p]) \cap \partial\mathrm{H} = \emptyset$. Let us assume that $g(\boldsymbol{\xi}') = 0$ for some $\boldsymbol{\xi}' \in \Xi((\eta_1^p, +\infty))$. Then $\boldsymbol{\xi}' \in \Lambda$, and hence we have $\xi_2' \le \xi_2^1 - (\lambda_1/\lambda_2)(\xi_1' - \xi_1^1)$ by noting $\lambda_2 = \eta_1^p - \eta_1^1 > 0$. Letting $\boldsymbol{\xi}'' = (\xi_1', \xi_2^1 - (\lambda_1/\lambda_2)(\xi_1' - \xi_1^1))$, we have $g(\boldsymbol{\xi}'') \ge 0$ by the monotonicity of $g$. Then $\boldsymbol{\eta}^p$, a convex combination of $\boldsymbol{\xi}''$ and $\boldsymbol{\xi}^1$, satisfies

$$g(\boldsymbol{\eta}^p) > \min\{g(\boldsymbol{\xi}''), g(\boldsymbol{\xi}^1)\} \ge 0,$$

which is a contradiction.
(ii) We have supposed that $\boldsymbol{\eta}^p$–$\boldsymbol{\eta}^1$ contains a point, say $\boldsymbol{\xi}'$, in $\Xi^\circ \setminus \mathrm{H}$. Taking $\delta = \xi_1' - \xi_1^1$ leads to the assertion. □

If (ii) holds in Lemma 3.3, we have to continue to search $\Xi((\xi_1^1, +\infty])$ for other active points, by using the procedure which will be developed in the next section.

### 3.2. Procedure for finding a locally best active point

Let us summarize the procedure. It receives an active point $\boldsymbol{\xi}^1$ such that $\boldsymbol{x}^*(\boldsymbol{\xi}^1)$ lies on some edge of $X$ containing a point in $X^\circ \setminus Y$, and then returns a number $\bar{\omega} \ge \xi_1^1$ and the best active point $\bar{\boldsymbol{\xi}}$ in the set $\Xi([\xi_1^1, \bar{\omega}])$. Let

$$M = \max\{\boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x} \in X\}.$$

**procedure** LOCAL($\boldsymbol{\xi}^1$);
**begin**
  $j := 1$ and $\bar{\boldsymbol{\xi}} := \boldsymbol{\xi}^j$;
  compute an optimal basis matrix $B_{j-1}$ of [P($\boldsymbol{\xi}^j$)] and the associated cell $\Phi_{j-1}$;
  let $\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^p$ denote the vertices of $\Phi_{j-1}$ in counterclockwise order from $\boldsymbol{\xi}^j$;
  **if** $g((1-\lambda)\boldsymbol{\xi}^j + \lambda\boldsymbol{\eta}^1) < 0$ for any $\lambda \in (0, 1]$ **then begin**
    move along $S(\Phi_{j-1})$ counterclockwise from $\boldsymbol{\eta}^1$ and choose as $\boldsymbol{\xi}^{j+1}$ the last point where the value of $g$ is nonpositive;
    let $B_j := B_{j-1}$, $\Phi_j := \Phi_{j-1}$ and $j := j + 1$;
    **if** $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^*(\boldsymbol{\xi}^j) > \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^*(\bar{\boldsymbol{\xi}})$ **then** update $\bar{\boldsymbol{\xi}} := \boldsymbol{\xi}^j$
  **end**;
  *stop* := *false*;
  **while** *stop* = *false* **do begin**
    choose a fundamental vector $\boldsymbol{e}^s$ such that $(\boldsymbol{e}^s)^{\mathrm{T}}B_{j-1}^{-1}(\tilde{\boldsymbol{b}} - \boldsymbol{e}^1\xi_1^j - \boldsymbol{e}^2\xi_2^j) = 0$;
    **if** $(\boldsymbol{e}^s)^{\mathrm{T}}B_{j-1}^{-1}N_{j-1} \ge \boldsymbol{0}$ for the nonbasic columns $N_{j-1}$ **then** *stop* := *true*
    **else begin**
      perform a dual pivot operation at the $s$th row in the dictionary with respect to $B_{j-1}$;
      let $\Phi_j$ denote the cell associated with the new basis $B_j$ and $\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^p$ the vertices of $\Phi_j$ in counterclockwise order from $\boldsymbol{\xi}^j$;

move along $S(\Phi_j)$ counterclockwise from $\boldsymbol{\eta}^1$ and choose as $\boldsymbol{\xi}^{j+1}$ the last point where the value of $g$ is nonpositive;
$j := j + 1$;
**if** $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^*(\boldsymbol{\xi}^j) > \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^*(\bar{\boldsymbol{\xi}})$ **then** update $\bar{\boldsymbol{\xi}} := \boldsymbol{\xi}^j$
**end**
**end**;
**if** $\eta_1^1 \leq \eta_1^p$ **then** return $(M, \bar{\boldsymbol{\xi}})$
**else** return $(\xi_1^j, \bar{\boldsymbol{\xi}})$
**end**;

LEMMA 3.4. *Under Assumption 2.1, procedure LOCAL terminates after finitely many iterations and returns a number* $\bar{\omega} \geq \xi_1^1$ *and an active point* $\bar{\boldsymbol{\xi}}$*, which provides the best incumbent* $\boldsymbol{x}^*(\bar{\boldsymbol{\xi}})$ *among all* $\boldsymbol{\xi} \in \Xi([\xi_1^1, \bar{\omega}]) \cap \partial\mathrm{H}$.

*Proof*. The procedure generates a sequence, $(\Phi_0,)$ $\boldsymbol{\xi}^1$, $\Phi_1$, $\boldsymbol{\xi}^2$, $\ldots$, $\Phi_{t-1}$, $\boldsymbol{\xi}^t$, until $(\boldsymbol{e}^s)^{\mathrm{T}}B_{t-1}^{-1}N_{t-1} \geq \mathbf{0}$ holds. Some $\Phi_j$s may appear more than once but no $\boldsymbol{\xi}^j$s do. By the convexity of $\Xi^\circ \setminus \mathrm{H}$, each edge of $\Phi_j$s can intersect $\partial\mathrm{H}$ not more than twice. This implies that each edge of $X$ contains two of $\boldsymbol{x}^*(\boldsymbol{\xi}^j)$s at most. Since $X$ has only a finite number of edges, the number of $\boldsymbol{\xi}^j$s is finite as well. We also see from Lemmas 3.2 and 3.3 that except for $\boldsymbol{\xi}^j$s no $\boldsymbol{\xi} \in \Xi([\xi_1^1, \bar{\omega}]) \cap \partial\mathrm{H}$ can provide an optimal solution to [P]. □

## 4. Search for a globally optimal solution to [P]

To generate a sequence of active points, procedure LOCAL requires a starting active point $\boldsymbol{\xi}^1$ such that $\boldsymbol{x}^*(\boldsymbol{\xi}^1) \in S(X)$. In this section, we will develop two procedures for supplying LOCAL with such an active point.

For an interval $\Omega$ let

$$X(\Omega) = X \cap \{\boldsymbol{x} \in X^\circ \mid \boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x} \in \Omega\},$$

like $\Xi(\Omega)$ defined in (3.2). We simply write $X(\omega)$ and $\Xi(\omega)$ for a degenerate interval $\Omega = [\omega, \omega]$. When searching for a starting active point, the following two parametric linear programs play important roles:

$$[\mathrm{C}(\omega)] \left| \begin{array}{ll} \text{maximize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & \boldsymbol{x} \in X(\omega), \end{array} \right. \qquad [\mathrm{D}(\omega)] \left| \begin{array}{ll} \text{minimize} & \boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & \boldsymbol{x} \in X(\omega). \end{array} \right.$$

Under condition (ii) of Assumption 2.1, both the problems have a unique optimal solution unless $X(\omega)$ is an empty set. Let $\boldsymbol{x}^{\mathrm{C}}(\omega)$ and $\boldsymbol{x}^{\mathrm{D}}(\omega)$ denote the optimal solutions to [C$(\omega)$] and [D$(\omega)$], respectively, and let

$$h_{\mathrm{C}}(\omega) = \boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x}^{\mathrm{C}}(\omega), \quad h_{\mathrm{D}}(\omega) = \boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x}^{\mathrm{D}}(\omega).$$

For any $\omega$ with $X(\omega) \neq \emptyset$, by the monotonicity of $g$, we have

$$g(\omega, h_{\mathrm{D}}(\omega)) \leq g(\omega, h_{\mathrm{C}}(\omega)).$$

LEMMA 4.1. *Suppose* $X(\omega) \neq \emptyset$. *Then*
*(i)* $\Xi(\omega) \cap \mathrm{H} = \emptyset$ *if* $g(\omega, h_{\mathrm{D}}(\omega)) > 0$*;*
*(ii)* $\Xi(\omega) \cap \partial\mathrm{H} \neq \emptyset$ *if* $g(\omega, h_{\mathrm{D}}(\omega)) \leq 0 \leq g(\omega, h_{\mathrm{C}}(\omega))$*;*
*(iii) no* $\boldsymbol{x}^*(\boldsymbol{\xi})$ *with* $\boldsymbol{\xi} \in \Xi(\omega)$ *is optimal to [P] if* $g(\omega, h_{\mathrm{C}}(\omega)) < 0$.

*Proof.* (i) For an arbitrary $\boldsymbol{\xi}' \in \Xi(\omega)$, there is some $\boldsymbol{x}' \in X(\omega)$ such that $\boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x}' = \xi_2'$. Since $h_{\mathrm{D}}(\omega) \leq \boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x}$ for all $\boldsymbol{x} \in X(\omega)$, we have $0 < g(\omega, h_{\mathrm{D}}(\omega)) \leq g(\omega, \boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x}') = g(\boldsymbol{\xi}')$ by the monotonicity of $g$. Hence, $\boldsymbol{\xi}'$ cannot be a point in H.
(ii) Obvious.
(iii) The optimal solution $\boldsymbol{x}^{\mathrm{C}}(\omega)$ to [C($\omega$)] satisfies $f(\boldsymbol{x}^{\mathrm{C}}(\omega)) = g(\omega, h_{\mathrm{C}}(\omega)) < 0$. Hence, $\boldsymbol{x}^{\mathrm{C}}(\omega)$ is feasible but not optimal to [P] by Theorem 2.1. Also, $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^{\mathrm{C}}(\omega) \geq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ for all $\boldsymbol{x} \in X(\omega)$, which implies that $\boldsymbol{x}^*(\boldsymbol{\xi})$ is not optimal to [P] for any $\boldsymbol{\xi} \in \Xi(\omega)$. □

Given a number $\omega^1$ such that $X(\omega^1) \neq \emptyset$, we can obtain an active point $\boldsymbol{\xi}^1$ with $\xi_1^1 > \omega^1$ by solving either [C($\omega$)] or [D($\omega$)] parametrically. We will show that no $\boldsymbol{\xi} \in \partial\mathrm{H}$ with $\xi_1 \in (\omega^1, \xi_1^1)$ provides an optimal solution to [P].

## 4.1. ROLE OF PROBLEM [C($\omega$)]

Let us consider

*case 4.1:* $X((\omega^1, \omega^1 + \delta]) \cap Y \neq \emptyset$ for any $\delta > 0$.

As will be seen later, the procedure below is applied to this case only when (iii) of Lemma 4.1 holds for $\omega = \omega^1$; therefore, we suppose here that $g(\omega^1, h_{\mathrm{C}}(\omega^1)) < 0$.

If we increase the value of $\omega$ from $\omega^1$ and solve [C($\omega$)] by using a parametric right-hand-side simplex algorithm, a sequence of intervals $[\omega^1, \omega^2], \ldots, [\omega^q, \omega^{q+1}]$, and associated bases $B_1', \ldots, B_q'$ will be generated, where $B_i' \in \mathbb{R}^{(m+1)\times(m+1)}$ is optimal to [C($\omega$)] for all $\omega \in [\omega^i, \omega^{i+1}]$ and $\omega^{q+1} = M$ (= $\max\{\boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x} \in X\}$). For each $i = 2, \ldots, q+1$, the optimal solution $\boldsymbol{x}^{\mathrm{C}}(\omega^i)$ is a vertex of $X$. There are two subcases under condition (iii) of Assumption 2.1:

$$g(\omega^i, h_{\mathrm{C}}(\omega^i)) < 0, \quad i = 2, \ldots, q+1; \tag{4.1}$$

$$g(\omega^i, h_{\mathrm{C}}(\omega^i)) < 0, \quad i = 2, \ldots, k\ (\leq q), \quad g(\omega^{k+1}, h_{\mathrm{C}}(\omega^{k+1})) > 0. \tag{4.2}$$

LEMMA 4.2. *In both (4.1) and (4.2), if*

$$g(\omega^i, h_{\mathrm{C}}(\omega^i)) < 0, \quad i = 2, \ldots, \ell\ (\leq q+1),$$

*then no* $\boldsymbol{x}^*(\boldsymbol{\xi})$ *with* $\boldsymbol{\xi} \in \Xi((\omega^1, \omega^\ell])$ *is optimal to [P].*

*Proof.* We see from Lemma 4.1 (iii) that no $\boldsymbol{x}^*(\boldsymbol{\xi})$ with $\xi_1 \in \{\omega^2, \ldots, \omega^\ell\}$ is optimal. If there is an active point $\boldsymbol{\xi}'$ with $\xi_1' \in (\omega^i, \omega^{i+1})$, then

$$\max\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^{\mathrm{C}}(\omega^i), \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^{\mathrm{C}}(\omega^{i+1})\} \geq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^{\mathrm{C}}(\xi_1') \geq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \quad \forall \boldsymbol{x} \in X(\xi_1').$$

Hence, no $\boldsymbol{\xi} \in \Xi((\omega^1, \omega^\ell])$ provides an optimal solution. □

If (4.2) holds, we choose as $\boldsymbol{\xi}^1$ an intersection of $(\omega^k, h_{\mathrm{C}}(\omega^k))$–$(\omega^{k+1}, h_{\mathrm{C}}(\omega^{k+1}))$ and $\partial\mathrm{H}$. By the convexity of $\Xi^\circ \setminus \mathrm{H}$, we can show that $\boldsymbol{\xi}^1$ is a unique intersection. From Lemma 4.1 (iii), no $\boldsymbol{\xi} \in \Xi((\omega^k, \xi_1^1))$ provides an optimal solution. We then have $\boldsymbol{x}^*(\boldsymbol{\xi}^1) = \boldsymbol{x}^{\mathrm{C}}(\xi_1^1)$, which lies on an edge $\boldsymbol{x}^{\mathrm{C}}(\omega^k)$–$\boldsymbol{x}^{\mathrm{C}}(\omega^{k+1})$ of $X$. Since one end of this edge is a point in $X^\circ \setminus Y$, procedure LOCAL can start from the active point $\boldsymbol{\xi}^1$.

The procedure for finding a starting active point in case 4.1 is summarized to the following:

**procedure** START1$(\omega^1)$;<br>
**begin**<br>
    $i$ := 1 and *stop* := *false*;<br>
    **while** *stop* = *false* **do begin**<br>
        compute a basis matrix $B_i'$ and a number $\omega^{i+1}$ such that $B_i'$ is optimal to [C$(\omega)$] for all $\omega \in [\omega^i, \omega^{i+1}]$;<br>
        **if** $g(\omega^{i+1}, h_{\mathrm{C}}(\omega^{i+1}) > 0$ **then begin**<br>
            let $\boldsymbol{\xi}^1$ be the intersection point of $(\omega^i, h_{\mathrm{C}}(\omega^i))$–$(\omega^{i+1}, h_{\mathrm{C}}(\omega^{i+1}))$ and $\partial$H;<br>
            *stop* := *true*<br>
        **end**<br>
        **else if** $\omega^{i+1} = M$ **then** $\boldsymbol{\xi}^1 := (\omega^{i+1}, h_{\mathrm{C}}(\omega^{i+1}))$ and *stop* := *true*<br>
        **else** $i := i + 1$<br>
    **end**;<br>
    return $\boldsymbol{\xi}^1$<br>
**end**;

### 4.2. ROLE OF PROBLEM [D$(\omega)$]

The rest to be considered is

*case 4.2:* $X((\omega^1, \omega^1 + \delta]) \cap Y = \emptyset$ for some $\delta > 0$.

Note from Lemma 3.3 that we have case 4.2 at $\omega^1 = \bar{\omega}$ if LOCAL returns $\bar{\omega} < M$.

As before, we solves [D$(\omega)$] for all $\omega \in [\omega^1, M]$ and generates a sequence of intervals $[\omega^1, \omega^2]$, ..., $[\omega^{q'}, \omega^{q'+1}]$, where $\omega^{q'+1} = M$. There are two subcases again:

$$g(\omega^i, h_{\mathrm{D}}(\omega^i)) > 0, \quad i = 2, \ldots, q' + 1; \tag{4.3}$$
$$g(\omega^i, h_{\mathrm{D}}(\omega^i)) > 0, \quad i = 2, \ldots, k \ (\leq q'), \quad g(\omega^{k+1}, h_{\mathrm{D}}(\omega^{k+1})) < 0. \tag{4.4}$$

LEMMA 4.3. *In both (4.3) and (4.4), if*

$$g(\omega^i, h_{\mathrm{D}}(\omega^i)) > 0, \quad i = 2, \ldots, \ell \ (\leq q' + 1),$$

*then* $\Xi((\omega^1, \omega^\ell]) \cap \mathrm{H} = \emptyset$.

*Proof.* For each $i = 2, \ldots, \ell$, the segment $(\omega^i, h_{\mathrm{D}}(\omega^i))$–$(\omega^{i+1}, h_{\mathrm{D}}(\omega^{i+1}))$ is included in the open convex set $\Xi^\circ \setminus$ H. Hence, $g(\omega, h_{\mathrm{D}}(\omega)) > 0$ for any $\omega \in [\omega^i, \omega^{i+1}]$; and the assertion follows from Lemma 4.1 (i). □

If (4.4) holds, we choose as $\boldsymbol{\xi}^1$ an intersection of $(\omega^k, h_{\mathrm{D}}(\omega^k))$–$(\omega^{k+1}, h_{\mathrm{D}}(\omega^{k+1}))$ and $\partial$H. Then we have $\boldsymbol{x}^*(\boldsymbol{\xi}^1) = \boldsymbol{x}^{\mathrm{D}}(\xi_1^1)$ lying on an edge $\boldsymbol{x}^{\mathrm{D}}(\omega^k)$–$\boldsymbol{x}^{\mathrm{D}}(\omega^{k+1})$ of $X$. As in case 4.1, the intersection $\boldsymbol{\xi}^1$ is unique, and no $\boldsymbol{\xi} \in \Xi((\omega^k, \xi_1^1))$ provides an optimal solution to [P].

The procedure for finding a starting active point in case 4.2 is as follows:

**procedure** START2$(\omega^1)$;
**begin**
    $i$ := 1 and *stop* := *false*;
    **while** *stop* = *false* **do begin**
        compute a basis matrix $B_i''$ and a number $\omega^{i+1}$ such that $B_i''$ is optimal to [D$(\omega)$] for all $\omega \in [\omega^i, \omega^{i+1}]$;
        **if** $g(\omega^{i+1}, h_{\mathrm{D}}(\omega^{i+1})) < 0$ **then begin**
            let $\boldsymbol{\xi}^1$ be the intersection of $(\omega^i, h_{\mathrm{D}}(\omega^i))$–$(\omega^{i+1}, h_{\mathrm{D}}(\omega^{i+1}))$ and $\partial$H;
            *stop* := *true*
        **end**
        **else if** $\omega^{i+1} = M$ **then** $\boldsymbol{\xi}^1 := (\omega^{i+1}, h_{\mathrm{D}}(\omega^{i+1}))$ and *stop* := *true*
        **else** $i := i + 1$
    **end**
    return $\boldsymbol{\xi}^1$
**end**;

4.3. Algorithm for finding an optimal solution to [P]

We are now ready to present the whole algorithm for computing a globally optimal solution $\boldsymbol{x}^*$ to [P]. It consists of procedure LOCAL in section 3.2 and the above two procedures.

**algorithm** GLOBAL;
**begin** {phase 1: find an initial active point $\boldsymbol{\xi}^1$}
    let $\boldsymbol{x}^1 := \arg\min\{\boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x} \in X\}$ and $\omega^1 := \boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x}^1$;
    **if** $g(\omega^1, \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^1) < 0$ **then** call START1$(\omega^1)$ to obtain $\boldsymbol{\xi}^1$
    **else** call START2$(\omega^1)$ to obtain $\boldsymbol{\xi}^1$;
    **if** $\xi_1^1 < M$ **then**
    **begin** {phase 2: find a globally optimal solution $\boldsymbol{x}^*$ to [P]}
        $\boldsymbol{\xi}^* := \boldsymbol{\xi}^1$ and *stop* := *false*;
        **while** *stop* = *false* **do begin**
            call LOCAL$(\boldsymbol{\xi}^1)$ to obtain $(\bar{\omega}, \bar{\boldsymbol{\xi}})$;
            **if** $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^*(\bar{\boldsymbol{\xi}}) > \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^*(\boldsymbol{\xi}^*)$ **then** update $\boldsymbol{\xi}^* := \bar{\boldsymbol{\xi}}$;

```
        if ω̄ = M then stop := true
        else begin
            call START2(ω̄) to obtain ξ^1;
            if ξ_1^1 = M then stop := true
        end
    end;
    x* := x*(ξ*)
  end
end;
```

We should note that procedure START1 is not called in phase 2. We see from Lemma 3.3 that case 4.2 occurs at $\omega^1 = \bar{\omega}$ whenever LOCAL returns $\bar{\omega} < M$. Therefore, after calling LOCAL, algorithm GLOBAL does not need START1 any more.

THEOREM 4.4. *Under Assumption 2.1, algorithm GLOBAL terminates after finitely many iterations and yields a globally optimal solution $\boldsymbol{x}^*$ of [P] if it exists.*

*Proof.* By Assumption 2.1 (ii), both procedures START1 and START2 are finite and either of them returns a point $\boldsymbol{\xi}^1$ in phase 1. From Lemmas 4.2 and 4.3, no $\boldsymbol{\xi}$ with $\xi_1 < \xi_1^1$ provides an optimal solution. If $\xi_1^1$ attains $M = \max\{\boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x} \in X\}$, then it must be yielded by START2$(\omega^1)$ under condition (2.3). In that case, $g(\omega, h_{\mathrm{D}}(\omega)) > 0$ for all $\omega \in [\omega^1, M]$ and hence [P] has no feasible solutions by Lemma 4.1 (i).

In phase 2, procedure LOCAL returns a number $\bar{\omega} \geq \xi_1^1$ and the best incumbent $\bar{\boldsymbol{\xi}}$ in $\Xi([\xi_1^1, \bar{\omega}]) \cap \partial$H. Unless $\bar{\omega}$ reaches $M$, case 4.2 occurs at $\omega^1 = \bar{\omega}$ and START2 is called to search $\Xi((\bar{\omega}, M])$ for an alternative $\boldsymbol{\xi}^1$ with $\xi_1^1 > \bar{\omega}$. In this way, LOCAL and START2 scan adjacent intervals covering H$^*$ alternately from $\xi_1 = \min\{\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x} \in X\}$ to $\xi_1 = M$ in the plane of $\Xi^\circ$. Some of the intervals scanned by LOCAL may be degenerate but none of those by START2 are. This, together with Lemma 3.4, implies that phase 2 of GLOBAL is finite and yields a globally optimal solution $\boldsymbol{x}^* = \boldsymbol{x}^*(\boldsymbol{\xi}^*)$ to [P]. □

### 4.4. NUMERICAL EXAMPLE

Before concluding this section, let us illustrate algorithm GLOBAL with the following small instance:

$$
\left|\begin{array}{ll}
\text{maximize} & x_3 \\
\text{subject to} & 5x_1 + 10x_2 + 5x_3 \leq 28 \\
& 8x_1 + 4x_2 + 5x_3 \leq 28 \\
& -130x_1 - 40x_2 + 90x_3 \leq 9 \\
& x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \\
& (3x_1 - x_2 + 3)(-x_1 + 3x_2 + 4) - 18 \leq 0.
\end{array}\right. \tag{4.5}
$$

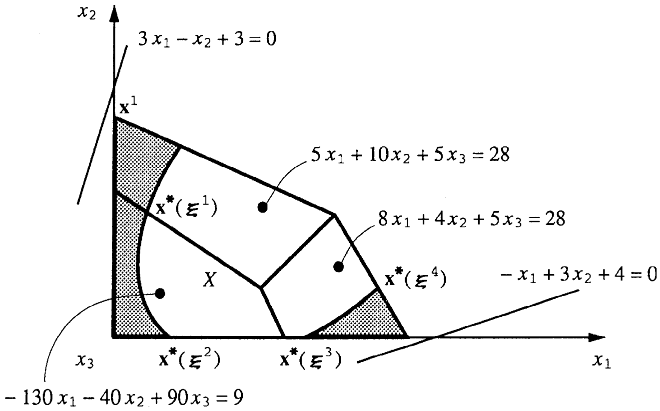


*Figure 1.* Three-dimensional example (4.5) of [P].

Let

$$X^\circ = \{(x_1, x_2, x_3)^{\mathrm{T}} \mid 3x_1 - x_2 + 3 > 0,\ -x_1 + 3x_2 + 4 > 0\}.$$

Then the product of two affine functions

$$f(\boldsymbol{x}) = (3x_1 - x_2 + 3)(-x_1 + 3x_2 + 4) - 18$$

is strictly quasiconcave (see e.g. [1]) and has rank-two monotonicity on $X^\circ$ with respect to $\boldsymbol{d}_1 = (3, -1, 0)^{\mathrm{T}}$ and $\boldsymbol{d}_2 = (-1, 3, 0)^{\mathrm{T}}$. We also see from Figure 1 that $X^\circ$ includes the polytope

$$X = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \begin{array}{ll} 5x_1 + 10x_2 + 5x_3 \le 28, & 8x_1 + 4x_2 + 5x_3 \le 28 \\ -130x_1 - 40x_2 + 90x_3 \le 9, & x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0 \end{array} \right\}.$$

The function $g$ in Lemma 2.2 is

$$g(\boldsymbol{\xi}) = (\xi_1 + 3)(\xi_2 + 4) - 18.$$

In phase 1, we first solve a linear program: minimize$\{3x_1 - x_2 \mid \boldsymbol{x} \in X\}$. Then we have $\boldsymbol{x}^1 = (0.000, 2.800, 0.000)^{\mathrm{T}}$ as its optimal solution. Since $f(\boldsymbol{x}^1) = -15.520 < 0$ and case 4.1 holds at $\omega^1 = \boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x}^1 = -2.800$, we need to solve the following problem in order to obtain an initial active point $\boldsymbol{\xi}^1$:

$$\left| \begin{array}{ll} \text{maximize} & x_3 \\ \text{subject to} & \boldsymbol{x} \in X \\ & 3x_1 - x_2 = \omega. \end{array} \right. \tag{4.6}$$

Procedure START1 solves (4.6) parametrically by increasing the value of $\omega$ from $-2.800$, and returns the first active point $\boldsymbol{\xi}^1$ after two pivot operations:

$$\boldsymbol{\xi}^1 = (-1.131, 5.631); \quad \boldsymbol{x}^*(\boldsymbol{\xi}^1) = (0.280, 1, 970, 1.379)^{\mathrm{T}}.$$

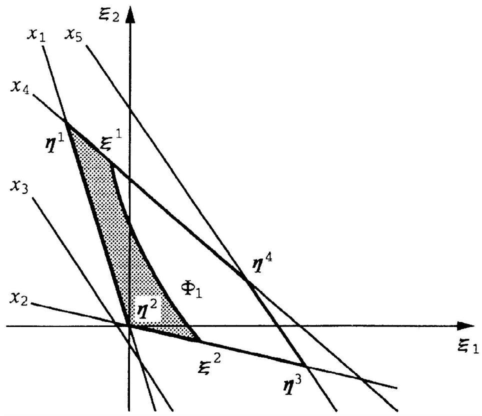


*Figure 2.* The cell $\Phi_1$ associated with the dictionary (4.8).

In phase 2, we solve the following problem as changing $\boldsymbol{\xi}$:

$$\left|\begin{array}{ll} \text{maximize} & x_3 \\ \text{subject to} & \boldsymbol{x} \in X \\ & 3x_1 - x_2 = \xi_1,\ -x_1 + 3x_2 = \xi_2. \end{array}\right. \tag{4.7}$$

The optimal dictionary of (4.7) at $\boldsymbol{\xi} = (-1.131, 5.631)$ is as follows:

$$\left|\begin{array}{l} x_2 = 0.000 + 0.125\xi_1 + 0.375\xi_2 \\ x_5 = 27.500 - 6.486\xi_1 - 4.236\xi_2 + 0.056x_6 \\ x_4 = 27.500 - 6.111\xi_1 - 6.111\xi_2 + 0.056x_6 \\ x_3 = 0.100 + 0.597\xi_1 + 0.347\xi_2 - 0.011x_6 \\ x_1 = 0.000 + 0.375\xi_1 + 0.125\xi_2 \\ z = 0.100 + 0.597\xi_1 + 0.347\xi_2 - 0.011x_6, \end{array}\right. \tag{4.8}$$

where $x_4$, $x_5$ and $x_6$ are slack variables. Hence, we define

$$\Omega_1 = \left\{ \boldsymbol{\xi} \in \mathbb{R}^2 \;\middle|\; \begin{array}{ll} 0.125\xi_1 + 0.375\xi_2 \geq 0, & 6.486\xi_1 + 4.236\xi_2 \leq 27.5 \\ 6.111\xi_1 + 6.111\xi_2 \leq 27.5, & 0.597\xi_1 + 0.347\xi_2 \geq -0.1 \\ 0.375\xi_1 + 0.125\xi_2 \geq 0 & \end{array} \right\}$$

(see Figure 2). We obtain the second active point $\boldsymbol{\xi}^2$ by computing the intersection of $g(\boldsymbol{\xi}) = 0$ and edge $\boldsymbol{\eta}^2$–$\boldsymbol{\eta}^3$ of $\Phi_1$. Performing a single dual pivot operation at the first row of (4.8) corresponding to $\boldsymbol{\eta}^2$–$\boldsymbol{\eta}^3$, we have

$$\boldsymbol{\xi}^2 = (3.000, -1.000); \quad \boldsymbol{x}^*(\boldsymbol{\xi}^2) = (1.000, 0.000, 1.544)^{\mathrm{T}}.$$

Since there is no active point $\boldsymbol{\xi}$ with $\xi_1 \in (3.000, 3.000 + \delta]$ for sufficiently small $\delta > 0$, case 4.2 holds and we have to solve

$$\left|\begin{array}{ll} \text{minimize} & -x_1 + 3x_2 \\ \text{subject to} & \boldsymbol{x} \in X \\ & 3x_1 - x_2 = \omega. \end{array}\right. \tag{4.9}$$

Procedure START2 solves (4.9) as increasing the value of $\omega$ from 3.000, and returns the third active point:

$$\boldsymbol{\xi}^3 = (6.000, -2.000); \quad \boldsymbol{x}^*(\boldsymbol{\xi}^3) = (2.000, 0.000, 2.400)^{\mathrm{T}}.$$

Starting from $\boldsymbol{\xi}^3$, procedure LOCAL again solves (4.7) and generate the last active point:

$$\boldsymbol{\xi}^4 = (9.857, -2.600); \quad \boldsymbol{x}^*(\boldsymbol{\xi}^4) = (3.371, 0.257, 0.000)^{\mathrm{T}}.$$

The maximum of $x_3$ is attained at $\boldsymbol{x}^*(\boldsymbol{\xi}^3)$.

## 5. Computational experiments

We will report computational results of testing algorithm GLOBAL. We coded GLOBAL and the algorithm proposed by Pferschy and Tuy [18] (abbr. P-T) in double precision C language, and tested them on a microSPARC II computer (70 MHz). The tolerance $\epsilon$ required by algorithm P-T for computing an $\epsilon$-optimal solution was fixed at $10^{-5}$.

The test problem was the following subclass of [P]:

$$\left|\begin{array}{ll} \text{maximize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} \leq \boldsymbol{b}, \ \boldsymbol{x} \geq \boldsymbol{0} \\ & (\boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x} - d_{10})(\boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x} - d_{20}) - d_{00} \leq 0, \end{array}\right. \tag{5.1}$$

where $\boldsymbol{c}$, $\boldsymbol{d}_i \in \mathbb{R}^n$ $(i = 1, 2)$, $d_{i0} \in \mathbb{R}$ $(i = 0, 1, 2)$, $\boldsymbol{b} \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$. To ensure that $f(\boldsymbol{x}) = (\boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x} - d_{10})(\boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x} - d_{20})$ is strictly quasiconcave and has rank-two monotonicity on $X = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \leq \boldsymbol{b}, \ \boldsymbol{x} \geq \boldsymbol{0}\}$, we put $-\boldsymbol{d}_1^{\mathrm{T}}\boldsymbol{x} \leq -d_{10} - 10^{-5}$ and $-\boldsymbol{d}_2^{\mathrm{T}}\boldsymbol{x} \leq -d_{20} - 10^{-5}$ in the first and second rows of $A\boldsymbol{x} \leq \boldsymbol{b}$, respectively. Components of $\boldsymbol{c}$, $\boldsymbol{d}_i$s and $A$ except the first two rows were drawn randomly from a uniform distribution over $[-1.000, 1.000]$, and those of $\boldsymbol{b}$ except the first two components and $d_{i0}$s were from $[0.000, 1.000]$. The size of problems ranged from $(m, n) = (100, 120)$ to $(250, 300)$. For each size we selected ten instances which were feasible and had no trivial solutions.

Table I shows the average performance of the two algorithms. For each size of $(m, n)$, the average number of pivot operations and the average CPU time in seconds (and their respective standard deviations in the brackets) needed to solve ten instances are listed. The column labeled Type 1 gives the number of primal pivot operations performed in phase 1; the column labeled Type 2 gives that of dual pivot operations performed by procedures START1 and START2; and the column labeled Type 3 gives that of dual pivot operations performed by procedure LOCAL.

We see from these results that algorithm GLOBAL is rather practical compared with algorithm P-T for randomly generated instances of (5.1). In particular, the total number of pivot operations required by GLOBAL is only about 25 % of that by

Table I. Computational results for (5.1).

| $m \times n$ | algorithm GLOBAL | | | | | algorithm P-T | |
|---|---|---|---|---|---|---|---|
| | # of pivots | | | | CPU time | # of pivots | CPU time |
| | type 1 | type 2 | type 3 | total | | | |
| 100 × 120 | 16.9 | 26.8 | 47.0 | 90.7 | 3.970 | 334.8 | 8.612 |
| | (7.2) | (19.0) | (28.2) | (22.1) | (1.425) | (260.3) | (6.922) |
| 150 × 120 | 21.3 | 25.3 | 61.8 | 108.4 | 7.960 | 456.2 | 19.888 |
| | (6.0) | (16.6) | (20.5) | (26.5) | (2.393) | (339.7) | (15.156) |
| 150 × 150 | 19.2 | 44.4 | 56.0 | 119.6 | 10.433 | 519.3 | 26.092 |
| | (10.3) | (34.4) | (22.5) | (26.9) | (3.856) | (574.1) | (30.198) |
| 150 × 180 | 38.1 | 51.1 | 71.0 | 160.2 | 13.662 | 774.5 | 41.903 |
| | (15.7) | (32.1) | (32.1) | (46.1) | (4.328) | (1015.9) | (56.338) |
| 200 × 180 | 28.3 | 64.1 | 57.4 | 149.8 | 19.208 | 402.1 | 31.275 |
| | (10.5) | (34.4) | (28.8) | (36.9) | (7.882) | (552.2) | (44.087) |
| 200 × 200 | 24.3 | 44.8 | 94.6 | 163.7 | 24.257 | 668.2 | 56.405 |
| | (10.5) | (38.7) | (37.7) | (32.2) | (6.684) | (827.7) | (71.312) |
| 200 × 220 | 23.1 | 45.2 | 81.4 | 149.7 | 22.617 | 586.8 | 49.437 |
| | (13.1) | (37.3) | (33.0) | (41.4) | (10.599) | (608.9) | (53.267) |
| 250 × 220 | 29.0 | 64.4 | 81.6 | 175.0 | 33.970 | 752.9 | 88.732 |
| | (9.6) | (46.5) | (43.1) | (61.5) | (14.929) | (770.5) | (92.617) |
| 250 × 250 | 37.9 | 90.7 | 92.0 | 220.6 | 45.585 | 936.4 | 117.728 |
| | (16.8) | (56.2) | (25.5) | (65.2) | (20.188) | (1141.2) | (147.679) |
| 250 × 300 | 41.2 | 70.8 | 91.0 | 203.0 | 45.193 | 836.5 | 118.395 |
| | (20.2) | (38.3) | (56.0) | (59.2) | (17.018) | (1085.0) | (159.672) |

algorithm P-T. Moreover, the variance of the former is far less than the latter. Since algorithm P-T discards local maxima by cutting off a portion of the feasible region, unfortunate cuts sometimes delay the convergence considerably. In contrast to this, algorithm GLOBAL uses no cuts and hence the convergence is relatively stable. It should also be emphasized that GLOBAL yields rigorous optimal solutions but not approximate ones.

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